Published by Institute of Physics Publishing for SISSA
Received: *April 28, 2008*
Accepted: *May 20, 2008*
Published: *June 5, 2008*

Lie 3-algebra and multiple M2-branes

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Abstract: Motivated by the recent proposal of an $N = 8$ supersymmetric action for multiple M2-branes, we study the Lie 3-algebra in detail. In particular, we focus on the fundamental identity and the relation with Nambu-Poisson bracket. Some new algebras not known in the literature are found. Next we consider cubic matrix representations of Lie 3-algebras. We show how to obtain higher dimensional representations by tensor products for a generic 3-algebra. A criterion of reducibility is presented. We also discuss the application of Lie 3-algebra to the membrane physics, including the Basu-Harvey equation and the Bagger-Lambert model.

Keywords: p-branes, M-Theory.

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Contents

1. Introduction

In the long history of the study of Nambu bracket [1], the relation with the supermembrane or M-theory has been giving the main motivation (see [2] for the references). There have been many attempts to quantize the classical Nambu bracket toward this direction. However, since the quantization is difficult and does not seem to be unique, we need to understand which properties are essential from the physical viewpoint.

Recently Bagger and Lambert [3 − 5] and Gustavsson [6] proposed a formalism of multiple M2-branes and it was found that the generalized Jacobi identity (or the fundamental identity) for Lie 3-algebra is essential to define the action with $\mathcal{N} = 8$ supersymmetry. It seems to give the desired principle of constructing quantum Nambu bracket which has been long sought for. So far the only explicit example of Lie 3-algebra ever considered for the Bagger-Lambert model is $\mathcal{A}_4$, the SO(4)-invariant algebra with 4 generators.[1] For a more concrete understanding of the Bagger-Lambert model, it is urgent to study more explicit examples of Lie 3-algebra. In the mathematical literature, the Lie 3-algebra (also known as Filippov algebra) is not new [9], and its structure has been studied to some extent. However, not only that the complete classification of the algebra does not exist, there are very few explicit examples in the literature.

In this paper, we first endeavor to find new examples of Lie 3-algebra (section 2). After a survey of the mathematical literature, especially the study of Nambu-Poisson bracket, interestingly, we successfully find several new examples (section 3). All the new examples have one important feature in common, namely that their metrics are not positive-definite. In this respect they are very different from $\mathcal{A}_4$. We also tried to search for solutions of the fundamental identity with positive-definite metrics by computer when the number of generators are small ($n = 5, 6, 7, 8$), and found that there are no algebras except for $\mathcal{A}_4$ and its direct sum. We are led to make the conjecture that there are no other 3-algebras with a positive definite metric. Generators of zero norm are almost ubiquitous in 3-algebras.

In section 4, we consider the problem of realizing Lie 3-algebras using cubic matrices. As an example, we consider cubic-matrix representations for $\mathcal{A}_4$, and try to develop a systematic method to generate higher dimensional representations. In the case of Lie algebra, a simple method to derive higher dimensional representations is to use the tensor product and then to decompose it into irreducible representations. Here we show that we can do similar construction of higher dimensional representations by tensor product. One can define the notion of irreducibility similarly, although we need to redefine the product of cubic matrices.

In section 5, we review Basu-Harvey equation, and demonstrate that its success in describing the configuration of multiple M2-branes ending on an M5-brane does not reply on the specific realization of the 3-algebra as it was originally considered. We only need the 3-algebra structure for the calculation. We also comment on its relation to the Bagger-Lambert model. A few comments about future directions are made in section 6.

In appendix A, we point out the relation between the fundamental identity and the Plücker relation. The latter appeared frequently in the literature of the exactly solvable system, matrix model and topological strings.

[1] See also Kawamura's work [7, 8] where the same algebra and its representation was studied.

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2. Lie n-algebra

2.1 Definitions

Lie n-algebra, also known as n-ary Lie algebra, or Filippov n-algebra [9], is a natural generalization of Lie algebra. For a linear space $\mathcal{V} = \{\sum_{a=1}^{\mathcal{D}} v_a T_a; v_a \in \mathbb{C}\}$ of dimension $\mathcal{D}$, a Lie n-algebra structure is defined by a multilinear map called Nambu bracket $[\cdot, \cdots, \cdot] : \mathcal{V}^{\otimes n} \to \mathcal{V}$ satisfying the following properties[2]

1. Skew-symmetry:
$$[A_{\sigma(1)}, \cdots, A_{\sigma(n)}] = (-1)^{|\sigma|}[A_1, \cdots, A_n]. \tag{2.1}$$

2. Fundamental identity:
$$[A_1, \cdots, A_{n-1}, [B_1, \cdots, B_n]] = \sum_{k=1}^{n} [B_1, \cdots, B_{k-1}, [A_1, \cdots, A_{n-1}, B_k], B_{k+1}, \cdots, B_n]. \tag{2.2}$$

The fundamental identity is also called the generalized Jacobi identity. It means that the bracket $[A_1, \cdots, A_{n-1}, \cdot]$ acts as a derivative on $\mathcal{V}$, and it may be used to represent a symmetry transformation.

In terms of the basis, n-algebra is expressed in terms of the (generalized) structure constants,

$$[T_{a_1}, \cdots, T_{a_n}] = i f_{a_1 \cdots a_n}{}^{b} T_b \tag{2.3}$$

The fundamental identity implies a bilinear relation the structure constants,

$$\sum_c f_{b_1 \cdots b_p}{}^{c} f_{a_1 \cdots a_{p-1} c}{}^{d} = \sum_i \sum_c f_{a_1 \cdots a_{p-1} b_i}{}^{c} f_{b_1 \cdots c \cdots b_p}{}^{d}. \tag{2.4}$$

One may introduce the inner product in the space of algebra $\mathcal{A}$ as a bilinear map from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{C}$

$$\langle T_a, T_b \rangle = h_{ab}. \tag{2.5}$$

We will refer to the symmetric tensor h_{ab} as the metric in the following. As a generalization of the Killing form in Lie algebra, we require that the metric is invariant under any transformation generated by the bracket $[T_{a_1}, \cdots, T_{a_{n-1}}, \cdot]$:

$$\langle [T_{a_1}, \cdots, T_{a_{n-1}}, T_b], T_c \rangle + \langle T_b, [T_{a_1}, \cdots, T_{a_{n-1}}, T_c] \rangle = 0. \tag{2.6}$$

This implies a relation for the structure constant

$$h_{cd} f_{a_1 \cdots a_{n-1} b}{}^{d} + h_{bd} f_{a_1 \cdots a_{n-1} c}{}^{d} = 0. \tag{2.7}$$

[2] In part of the literature [10], the fundamental identity (2.2) is replaced by a weaker (skew-symmetrized) version, and thus the definition of Lie n-algebra is ambiguous. The definition we consider here is more closely related to the physical applications we will consider below. See also [11] for various aspects of the classical and quantum Nambu bracket.

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Therefore the tensor

$$f_{a_1\cdots a_n} \equiv f_{a_1\cdots a_{n-1}}{}^b h_{ba_n} \tag{2.8}$$

is totally antisymmetrized.

For applications to physics, it is very important to have a nontrivial metric h_{ab} in order to write down a Lagrangian or physical observables which are invariant under transformations defined by n-brackets.

Another mathematical structure of physical importance is Hermitian conjugation. A natural definition of the Hermitian conjugate of an n-bracket is

$$[A_1, \cdots, A_n]^\dagger = [A_n^\dagger, \cdots, A_1^\dagger]. \tag{2.9}$$

This relation determines the reality of structure constants. For the usual Lie algebra, if we choose the generators to be Hermitian, the structure constants $f_{ab}{}^c$ are real numbers, and if the generators are anti-Hermitian, the structure constants are imaginary. This is not the case for 3-brackets. The structure constants are always imaginary when the generators are all Hermitian or all anti-Hermitian. In general, for n-brackets, the structure constants are real if $n = 0, 1 \pmod 4$, and imaginary if $n = 2, 3 \pmod 4$ for Hermitian generators. The structure constants are multiplied by a factor of $\pm i$ when we replace Hermitian generators by anti-Hermitian ones only for even n.

From now on we will focus on the case of $n = 3$. Explicitly, for 3-algebra the fundamental identity (2.2) is

$$[A_1, A_2, [B_1, B_2, B_3]] = [[A_1, A_2, B_1], B_2, B_3] + [B_1, [A_1, A_2, B_2], B_3] + [B_1, B_2, [A_1, A_2, B_3]]. \tag{2.10}$$

In terms of the structure constant, the fundamental identity is

$$\sum_i f_{cde}{}^i f_{abi}{}^j = \sum_i \left(f_{abc}{}^i f_{ide}{}^j + f_{abd}{}^i f_{cie}{}^j + f_{abe}{}^i f_{cdi}{}^j \right) . \tag{2.11}$$

One of the important questions is how to classify the solutions of the fundamental identity (2.11) (or more generally (2.4)). The trivial solution is to put all structure constants zero $f_{abc}{}^d = 0$. The simplest nontrivial solution which satisfy the fundamental identity (2.11) of 3-algebra starts from $\mathcal{D} = 4$,

$$[T_a, T_b, T_c] = i\epsilon_{abcd} T_d, \qquad (a, b, c, d = 1, 2, 3, 4), \tag{2.12}$$

and the metric is fixed by the requirement of invariance (2.7) to be

$$h_{ab} = \delta_{ab} \tag{2.13}$$

up to an overall constant factor. Compared with the formula in some literature, we have an extra factor of i on the right hand side of (2.12) due to our convention of the Nambu bracket's Hermiticity (2.9).

This algebra is invariant under SO(4), and will be denoted as $\mathcal{A}_4$. The structure constant is given by the totally antisymmetrized epsilon tensor $f_{abc}{}^d = i\,\epsilon_{abcd}$. In general, for any n, the fundamental identity (2.4) is solved by the epsilon tensor in $\mathcal{D} = n + 1$,

$$f_{a_1\cdots a_n}{}^b = i\,\epsilon_{a_1\cdots a_n b}\,, \tag{2.14}$$

with the metric (2.13).

From these algebras, one may obtain higher rank algebras by direct sum as usual. For $n = 3$ case, the algebra $\mathcal{A}_4 \oplus \cdots \oplus \mathcal{A}_4$ (p-times) with $\mathcal{D} = 4p$ is written as,

$$[T_a^{(\alpha)}, T_b^{(\beta)}, T_c^{(\gamma)}] = i\epsilon_{abcd}\delta_{\alpha\beta\gamma\delta}T_d^{(\delta)}, \qquad (a,b,c,d = 1,2,3,4, \quad \alpha,\beta,\gamma,\delta = 1,\cdots,p)\,, \tag{2.15}$$

where $\delta_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta}$.

A nontrivial question is whether there exists any 3-algebra which can not be reduced to the direct sums of the algebra $\mathcal{A}_4$, up to a direct sum with a trivial algebra. For $n = 3$, one may directly solve the fundamental identity by computer for lower dimensions $\mathcal{D}$. We have examined the cases $\mathcal{D} = 5, 6, 7, 8$ with the assumption that the metric h_{ab} is invertible and can be set to δ_{ab} after the change of basis. In this case the structure constant ${f_{abc}}^d$ can be identified with totally anti-symmetric four tensor f_{abcd}.

For $\mathcal{D} = 5, 6$, one can solve directly the fundamental identity algebraically by computer. For $\mathcal{D} = 7, 8$, we assume the coefficients f_{abcd} are integer and $|f_{abcd}| \le 3$ and scanned all possible combinations. After all, the solutions can always be reduced to $\mathcal{A}_4$ up to a direct sum with a trivial algebra, or $\mathcal{A}_4 \oplus \mathcal{A}_4$ ($\mathcal{D} = 8$) after a change of basis.[3] This observation suggests that the Lie n-algebra for $n > 2$ is very limited.

Actually there is an interesting relation between the fundamental identity and the Plücker relation (for the Grassmaniann manifold), which will be explained in the appendix. It automatically tells us that the epsilon tensor is the solution of the fundamental identity for Lie n-algebra in general. At the same time, it also implies that to find other solutions are very difficult.

While very little is known about explicit nontrivial examples of the n-algebra, its correspondence with Nambu-Poisson brackets given in section 2.2 is very helpful.

If the metric is not invertible, it becomes possible to construct Lie 3-algebra other than the direct sum of $\mathcal{A}_4$. We will construct some examples in section 3.

2.2 Review of Nambu-Poisson brackets

Let $\mathcal{M}_d$ be a manifold of d dimensions, and $C(\mathcal{M}_d)$ its algebra of functions. A Nambu-Poisson bracket is a multi-linear map from $C(\mathcal{M}_d)^{\otimes 3}$ to $C(\mathcal{M}_d)$ that satisfies the following conditions [13]:

1. Skew-symmetry:

$$\{f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}\} = (-1)^{|\sigma|}\{f_1, f_2, f_3\}. \tag{2.17}$$

[3] One of the failed examples is,

$$\sum_{a,b,c,d=1}^{7} f_{abcd}\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \wedge \mathbf{e}_d = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 - \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_5 \wedge \mathbf{e}_7 + \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7 + \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7 + \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 \wedge \mathbf{e}_7 + \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \tag{2.16}$$

for $\mathcal{D} = 7$. This is the Hodge dual of G_2-invariant 3-form. It was also mentioned in [12].

2. Leibniz rule:
$$\{f_1, f_2, gh\} = \{f_1, f_2, g\}h + g\{f_1, f_2, h\}. \tag{2.18}$$

3. Fundamental identity:
$$\{g, h, \{f_1, f_2, f_3\}\} = \{\{g, h, f_1\}, f_2, f_3\} + \{f_1, \{g, h, f_2\}, f_3\} + \{f_1, f_2, \{g, h, f_3\}\}. \tag{2.19}$$

The prototype of a Nambu-Poisson bracket is the Jacobian determinant for 3 variables $x_i (i = 1, 2, 3)$
$$\{f_1, f_2, f_3\} = \epsilon_{ijk}\partial_i f_1 \partial_j f_2 \partial_k f_3. \tag{2.20}$$
where $i, j, k = 1, 2, 3$. This is the classical Nambu bracket. More general Nambu-Poisson bracket can be written in terms of the local coordinates as,
$$\{f_1, f_2, f_3\} = \sum_{i_1<i_2<i_3} \sum_{\sigma\in S_3} (-1)^\sigma P_{i_1 i_2 i_3}(x)\partial_{i_{\sigma(1)}} f_1 \partial_{i_{\sigma(2)}} f_2 \partial_{i_{\sigma(3)}} f_3. \tag{2.21}$$

It is proved that one can always choose coordinates such that any Nambu-Poisson bracket is locally just a Jacobian determinant [14]. Locally we can choose coordinates such that
$$\{f, g, h\} = \epsilon^{ijk}\partial_i f\ \partial_j g\ \partial_k h, \tag{2.22}$$
where $i, j, k = 1, 2, 3$, and $dx_1 dx_2 dx_3$ defines a local expression of the volume form. As a result, it is straightforward to check that the Nambu-Poisson bracket can be used to generate volume-preserving diffeomorphisms on a function f
$$\delta f = \{g_1, g_2, f\} \tag{2.23}$$
specified by two functions g_1 and g_2.

A Nambu-Poisson algebra is also an infinite dimensional Lie 3-algebra. For a 3-manifold on which the Nambu-Poisson bracket is everywhere non-vanishing, it is natural to use the volume form picked by the bracket to define an integral $\int_{\mathcal{M}}$, and then the metric can be defined by
$$\langle f, g\rangle = \int_{\mathcal{M}} fg. \tag{2.24}$$
Symmetries of the algebra are then automatically preserved by the metric.

The notion of Nambu-Poisson brackets can be naturally generalized to brackets of order n, as a map from $C(\mathcal{M}_d)^{\otimes n}$ to $C(\mathcal{M}_d)$. The fundamental identity for Nambu-Poisson brackets of order n is
$$\{f_1, \cdots, f_{n-1}, \{g_1, \cdots, g_n\}\} = \sum_{k=1}^{n} \{g_1, \cdots, g_{k-1}, \{f_1, \cdots, f_{n-1}, g_k\}, g_{k+1}, \cdots, g_n\}. \tag{2.25}$$
Both the Leibniz rule and the fundamental identity indicate that it is natural to think of
$$\{f_1, \cdots, f_{n-1}, \ \cdot\ \} : C(\mathcal{M}_d) \to C(\mathcal{M}_d) \tag{2.26}$$

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as a derivative on functions.

Each Nambu-Poisson bracket of order n corresponds to a Nambu-Poisson tensor field P through the relation

$$\{f_1, \cdots, f_n\} = P(df_1, \cdots, df_n), \tag{2.27}$$
$$P = \sum_{i_1<\cdots<i_n} P_{i_1\cdots i_n}(x)\partial_{i_1} \wedge \cdots \wedge \partial_{i_n} . \tag{2.28}$$

The theorem mentioned above can also be generalized to brackets of order n, which means that any Nambu-Poisson tensor field P is decomposable, i.e., one can express P as

$$P = V_1 \wedge \cdots \wedge V_n \tag{2.29}$$

for n-vector fields V_i. For a review of Nambu-Poisson brackets see, e.g. [15].

Let us now focus on the case $n = 3$. When all the coefficients of the Nambu-Poisson tensor field are linear in x, that is, $P_{i_1 i_2 i_3}(x) = \sum_j f_{i_1 i_2 i_3}{}^j x_j$ for constant $f_{i_1 i_2 i_3}{}^j$, we call the bracket a linear Nambu-Poisson bracket, and it takes the form of a Lie 3-algebra on the coordinates

$$\{x_i, x_j, x_k\} = \sum_l f_{ijk}{}^l x_l. \tag{2.30}$$

Apparently, a linear Nambu-Poisson bracket is also a Lie 3-algebra when we restrict ourselves to linear functions of the coordinates x_i. We have to be careful, however, in that the reverse is not true, as they also have some differences. For the Nambu-Poisson bracket, one may change the coordinates by a general coordinate transformation. On the other hand, for Lie 3-algebra, we only allow linear transformations of the basis. Since the requirement of Leibniz rule for the Nambu-Poisson bracket is not imposed on a Lie 3-algebra, we expect that only a small fraction of Lie 3-algebras are also linear Nambu-Poisson algebras. In particular, we do not expect that the Nambu bracket of a generic Lie 3-algebra be decomposable.

It has been shown that any linear Nambu-Poisson tensor of order n on a linear space V_d can be put in one of the following forms by choosing a suitable basis of V_d [16]:

1. Type I:

$$P_{(r,s)} = \sum_{j=1}^{r+1} \pm x_j \partial_1 \wedge \cdots \wedge \partial_{j-1} \wedge \partial_{j+1} \wedge \cdots \wedge \partial_{n+1} \tag{2.31}$$
$$+ \sum_{j=1}^{s} \pm x_{n+j+1} \partial_1 \wedge \cdots \wedge \partial_{r+j} \wedge \partial_{r+j+2} \wedge \cdots \wedge \partial_{n+1},$$

where $-1 \le r \le n$, $0 \le s \le \min(d-n-1, n-r)$. Explicitly, we have

$$\{x_1, \cdots, x_{j-1}, x_j, \cdots, x_{n+1}\} = \begin{cases} \pm x_j, & 1 \le j \le r+1, \\ \pm x_{j-r+3}, & r+2 \le j \le r+s+1, \\ 0, & r+s+2 \le j \le d. \end{cases} \tag{2.32}$$

2. Type II:

$$P = \partial_1 \wedge \cdots \wedge \partial_{n-1} \wedge \left(\sum_{i,j=n}^{d} a_{ij} x_i \partial_j \right). \tag{2.33}$$

In other words,

$$\{x_1, \cdots, x_{n-1}, x_j\} = \sum_{i=n}^{d} a_{ij} x_i, \qquad j = n, \cdots, d. \tag{2.34}$$

Here the choice of coordinates is made such that the Nambu-Poisson tensor field is linear, instead of trying to make its decomposability manifest. When we interpret these brackets as Nambu brackets on the linear space generated by $\{x_i\}$, we are no longer allowed to make general coordinate transformations on the generators x_i, and the decomposability of the Nambu-Poisson tensor field is no longer relevant.

3. Examples of Lie 3-algebra

We already know a few examples of Lie 3-algebra which satisfies the fundamental identity.

- A trivial algebra is one for which the Nambu bracket is always 0.
- The 4-generator algebra with SO(4) symmetry $\mathcal{A}_4$.
- Direct sums of an arbitrary number of copies of $\mathcal{A}_4$ and a trivial algebra.
- All Nambu-Poisson brackets on $C(\mathcal{M}_d)$ are of course also Nambu brackets on the infinite dimensional linear space $C(\mathcal{M}_d)$.

In the following, we list a few more examples of Lie 3-algebra. In contrast with previous studies on this problem, we put relatively more emphasis on the metric, which is crucial for writing down an invariant observable or Lagrangian.[4] Besides $\mathcal{A}_4$, the only well known example of 3-algebra is the class constructed in [19]. However, as we will show below in section 3.3, the invariant metric is almost trivial in those cases.

3.1 Linear Nambu-Poisson bracket: type I

First, since any linear Nambu-Poisson bracket is also a Lie 3-algebra, the classification of the last subsection gives type I and type II algebras.

A type I linear Nambu-Poisson bracket $P_{(r,s)}$ (2.31), (2.32) is labeled by a pair of integers (r, s). $P_{(3,0)}$ in (2.31) with plus signs for $n = 3$ gives $\mathcal{A}_4$ algebra. For other values of (r, s), $P_{(r,s)}$ gives a new algebra.

For example, $P_{(-1,4)}$ defines an algebra with 8 generators (apart from direct sum with a trivial algebra)

$$[T_2, T_3, T_4] = \pm T_5, \quad [T_1, T_3, T_4] = \pm T_6, \quad [T_1, T_2, T_4] = \pm T_7, \quad [T_1, T_2, T_3] = \pm T_8. \tag{3.1}$$

[4] However, [18] suggests that we study the Bagger-Lambert model only at the level of equations of motion, which can be described without a metric.

Without loss of generality, we can take all plus signs above, and an invariant metric is given by

$$h_{15} = -h_{26} = h_{37} = -h_{48} = K \tag{3.2}$$

for some constant K. The metric is thus non-degenerate with the signature $(++++----)$.

Another example is $P_{(1,1)}$, which is defined by

$$[T_2, T_3, T_4] = -T_1, \quad [T_1, T_3, T_4] = \epsilon T_2, \quad [T_1, T_2, T_4] = T_5, \quad [T_1, T_2, T_3] = T_6, \tag{3.3}$$

where we have fixed the signs except $\epsilon = \pm 1$ by convention. The invariant metric is given by

$$h_{11} = \epsilon h_{22} = h_{35} = -h_{46} = 1, \tag{3.4}$$

while other components of h vanish.

3.2 Linear Nambu-Poisson bracket: type II

The linear Nambu-Poisson algebra of type II (2.33), (2.34) for arbitrary constant matrix a_{ij} has the Nambu bracket

$$[T_1, T_2, T_j] = \sum_{i=3}^{d} a_{ij} T_i \quad (j = 3, \cdots, d)\,. \tag{3.5}$$

The invariance of the metric implies that

$$h_{i1} = h_{i2} = \sum_{i=3}^{d} h_{ji} a_{ik} = 0 \tag{3.6}$$

for $i, j, k = 3, \cdots, d$. Thus $a = 0$ if h is invertible. Conversely, if a is invertible then $h_{ij} = 0$ for $i, j = 3, \cdots, d$. As T_1 and T_2 do not appear on the right hand side of the Nambu bracket, there is no constraint on h_{11}, h_{12} or h_{22}.

As Nambu-Poisson brackets, we can extend the 3-algebra on the space of linear functions $\mathcal{V} = \{\sum_{i=1}^{d} a_i T_i\}$ to all polynomials of T_i's. The product of T_i's defines a commutative algebra.

3.3 One-generator extension of a Lie algebra

In addition, we may construct other examples. For a given Lie algebra $\mathcal{G}$ with generators T_a and structure constants $f_{ab}{}^{c}$, we can introduce a new element T_0 and define a Lie 3-algebra by [20]

$$[T_0, T_a, T_b] = f_{ab}{}^{c} T_c, \tag{3.7}$$

$$[T_a, T_b, T_c] = 0 \tag{3.8}$$

for $a, b, c = 1, \cdots, \dim \mathcal{G}$. For a simple Lie algebra $\mathcal{G}$, the invariance of the metric demands that

$$\langle [T_0, T_a, T_b], T_c \rangle + \langle T_b, [T_0, T_a, T_c] \rangle = 0 \quad \Rightarrow \quad f_{ab}{}^{d} h_{dc} + f_{ac}{}^{d} h_{db} = 0. \tag{3.9}$$

This suggests that h_{ab} should be proportional to the Killing form of $\mathcal{G}$. However, the invariance conditions also include

$$\langle [T_a, T_b, T_c], T_0\rangle + \langle T_c, [T_a, T_b, T_0]\rangle = 0 \Rightarrow f_{ab}{}^d h_{dc} = 0,$$
$$\langle [T_a, T_b, T_0], T_0\rangle + \langle T_0, [T_a, T_b, T_0]\rangle = 0 \Rightarrow h_{c0} = 0. \tag{3.10}$$

Therefore, we can not use the Killing form of the Lie algebra $\mathcal{G}$ as h_{ab}, but instead the metric should be taken as

$$h_{ab} = h_{0a} = 0, \qquad h_{00} = K, \qquad a, b = 1, \cdots, \dim \mathcal{G}, \tag{3.11}$$

where K is an arbitrary constant.

If the Lie algebra $\mathcal{G}$ can be realized as a matrix algebra, this 3-algebra can also be extended to polynomials of T_a's. (That is, we extend the Lie algebra $\mathcal{G}$ to its universal enveloping algebra.) We can define the Nambu bracket by

$$[T_0, A, B] = [A, B] \equiv AB - BA, \qquad [A, B, C] = 0, \tag{3.12}$$

where A, B, C are elements of the matrix algebra. The Leibniz rule follows from this definition[5]

$$[T_0, A, BC] = [T_0, A, B]C + B[T_0, A, C]. \tag{3.13}$$

However, it is not possible for the Leibniz rule to apply to products involving T_0.

This 3-algebra has a close connection with the Nambu bracket defined in [19]. For a matrix algebra, the Nambu bracket in [19] is defined as

$$[A, B, C] = \mathrm{tr}(A)[B, C] + \mathrm{tr}(B)[C, A] + \mathrm{tr}(C)[A, B]. \tag{3.14}$$

This Nambu bracket is automatically skew-symmetric and satisfies the fundamental identity. For a matrix algebra, we can choose the basis of generators such that there is only one generator, the identity I, that has a non-vanishing trace. Denoting $T_0 = I/\mathrm{tr}(I)$, and the rest of the generators as T_a ($a \neq 0$), the Nambu bracket is precisely given by (3.7) and (3.8). Thus we see that the Nambu bracket of [19] is equivalent to the 3-algebra in this subsection for the case when $\mathcal{G}$ is a matrix algebra of traceless matrices.

3.4 A truncation of Nambu-Poisson structure on S^3

The classical Nambu bracket

$$\{f_1, f_2, f_3\} = x_i\, \epsilon_{ijkl}\, \partial_j f_1\, \partial_k f_2\, \partial_l f_3 \tag{3.15}$$

defines a Nambu-Poisson bracket with SO(4) symmetry on the space of all polynomials of $\{x_i : i = 1, \cdots, 4\}$ to all order. Based on this we define a Nambu bracket which is restricted to polynomials of order no larger than N as

$$[X_{i_1\cdots i_l}, X_{j_1\cdots j_m}, X_{k_1\cdots k_n}] = \begin{cases} \{X_{i_1\cdots i_l}, X_{j_1\cdots j_m}, X_{k_1\cdots k_n}\}, & l+m+n-2 \leq N, \\ 0, & l+m+n-2 > N, \end{cases} \tag{3.16}$$

[5]Note that here the ordering of the product on the right hand side is important, unlike the case of a Nambu-Poisson algebra.

where the generators X are monomials of order $l \leq N$

$$X_{i_1 \cdots i_l} = x_{i_1} \cdots x_{i_l}. \tag{3.17}$$

The case with $N = 1$ is precisely $\mathcal{A}_4$. As $N \to \infty$, this algebra approaches to a classical Nambu-Poisson structure on $C(\mathbb{R}^4)$.

As the Nambu-Poisson algebra (3.15) is known to observe the fundamental identity, we only need to check that the truncation rule is compatible with it. Note that each term in the fundamental identity is of the form $[A_1, A_2, [A_3, A_4, A_5]]$. Let each A_i to be a monomial of order a_i. Then this term is truncated to zero if $a_3 + a_4 + a_5 - 2 > N$ so that $[A_3, A_4, A_5]$ is truncated to zero, or if $a_1 + \cdots + a_5 - 4 > N$ so that the outer bracket vanishes. However, since a monomial is at least of order 1,[6] we always have

$$a_1 + a_2 + a_3 + a_4 + a_5 - 4 \geq a_3 + a_4 + a_5 - 2, \tag{3.18}$$

and hence the necessary and sufficient condition for truncation for every term in the fundamental identity is the same

$$\sum_{i=1}^{5} a_i - 4 > N. \tag{3.19}$$

Thus the fundamental identity is preserved by the truncation rule.

We can also try to define multiplication by truncating the products of monomials as

$$X_{i_1 \cdots i_l} \cdot X_{j_1 \cdots j_m} = \begin{cases} X_{i_1 \cdots i_l j_1 \cdots j_m}, & l + m \leq N \\ 0, & l + m > N. \end{cases} \tag{3.20}$$

Again, one can check that the Leibniz rule, which is known to hold for the case $N = \infty$, is compatible with the truncation of products at finite N. Indeed, every term in the Leibniz rule condition

$$[A_1, A_2, A_3 A_4] = [A_1, A_2, A_3] A_4 + [A_1, A_2, A_4] A_3 \tag{3.21}$$

is truncated if and only if

$$a_1 + a_2 + a_3 + a_4 - 2 > N. \tag{3.22}$$

To define the metric, it is natural to use the integration over the underlying manifold. Decomposing the integration over the space of x_i into the radial part and the integration over S^3, we define the metric as

$$\langle A_1, A_2 \rangle = \int_{S^3} d^3\Omega \int_0^\infty dr \rho(r) A_1 \cdot A_2, \tag{3.23}$$

where we introduced a distribution $\rho(r)$ so that the integrals converge for polynomials of x_i. If we are considering the Nambu structure on a truncated set of functions on S^3 of radius R, we should take $\rho(r) = \delta(r - R)$.

[6]If one of the entries is of order 0 (that is, it is a constant), the Nambu bracket vanishes identically.

Roughly speaking, treating x_i as coordinates on S^3 is equivalent to imposing the constraint

$$\sum_{i=1}^{4} x_i^2 = 1 \tag{3.24}$$

on the algebra of polynomials of x_i's. Since $\sum_i x_i^2$ is a central element in the 3-algebra, i.e.

$$[\sum_i x_i^2, X_{i_1\cdots i_l}, X_{j_1\cdots j_m}] = 0, \tag{3.25}$$

this constraint is consistent with the Nambu structure. However, the constraint is not compatible with the truncation rule for the Nambu bracket (3.16) or the product (3.20). Thus we should not impose the constraint except when we compute the metric. The metric of $\langle A, B\rangle$ should be computed by first multiplying $A \cdot B$ with the truncation (3.20), and then treating the product as a classical function on S^3 and integrate.

It is easy to see the the metric defined this way is not positive definite. Consider the norm of $A = x_1 - a x_1^m$, where m is an odd number between $N/2 + 1$ and $N - 1$. Its norm is

$$\langle A, A\rangle = \int_{S^3} x_1^2 - 2a \int_{S^3} x_1^{m+1}, \tag{3.26}$$

where the term $\langle x_1^m, x_1^m\rangle$ is absent because $x_1^m \cdot x_1^m = 0$ according to (3.20). While both terms on the right hand side are non-zero, one can choose a to be sufficiently large so that the norm is negative.

3.5 An Extension of $\mathcal{A}_4$

An algebra with $4(N + 1)$ generators $\{T_i^{(a)} : a = 0, \cdots , N, i = 1, \cdots , 4\}$ can be defined by

$$[T_i^{(a)}, T_j^{(b)}, T_k^{(c)}] = \begin{cases} \epsilon_{ijkl}\, T_l^{(a+b+c)}, & a + b + c \leq N, \\ 0, & a + b + c > N. \end{cases} \tag{3.27}$$

To check that the Nambu bracket (3.27) preserves the fundamental identity, we only need to check that the truncation rule is compatible with the fundamental identity, since this bracket is essentially just a grading of direct sums of $\mathcal{A}_4$. For a term in the fundamental identity

$$[T_i^{(a)}, T_j^{(b)}, [T_k^{(c)}, T_l^{(d)}, T_m^{(e)}]], \tag{3.28}$$

we note that it is truncated if $c + d + e > N_2$ (so that the inner bracket is zero), or if $a+b+c+d+e > N$ (so that the outer bracket is zero). However, since $a, b \geq 0$, we always have $a + b + c + d + e > c + d + e$, and thus the necessary and sufficient condition for this term to be truncated to zero is just $a + b + c + d + e > N$. Since this condition is the same for all terms in the fundamental identity, the fundamental identity is preserved.

One can further extend the 3-algebra form the linear space spanned by $T_i^{(a)}$'s to polynomials of the generators truncated at order N. Let

$$T_i^{(a)} T_j^{(b)} = \begin{cases} T_j^{(b)} T_i^{(a)}, & a + b \leq N, \\ 0, & a + b > N. \end{cases} \tag{3.29}$$

The space of polynomials of $T_i^{(a)}$'s is thus spanned by the monomials $\{T_{i_1}^{(a_1)} \cdots T_{i_k}^{(a_k)} : \sum_{r=1}^k a_r \leq N\}$. The Nambu bracket on this space can be defined by imposing the Leibniz rule

$$[A^{(a)}, B^{(b)}, C^{(c)} D^{(d)}] = [A^{(a)}, B^{(b)}, C^{(c)}] D^{(d)} + [A^{(a)}, B^{(b)}, D^{(d)}] C^{(c)}, \tag{3.30}$$

where $A^{(a)}$ is a monomial $T_{i_1}^{(a_1)} \cdots T_{i_k}^{(a_k)}$ of level $\sum_{r=1}^k a_r = a$, etc. Note that the truncation rule of every term above is that each term vanishes if and only if $a + b + c + d \geq N$.

For a given function $f(a)$ with the property

$$f(a) = 0 \quad \text{for} \quad a > N, \tag{3.31}$$

the invariant metric can be defined as

$$\langle T_i^{(a)}, T_j^{(b)} \rangle = f(a+b)\delta_{ij} \qquad \text{for} \quad a, b = 0, \cdots, N, \;\; i, j = 1, \cdots, 4. \tag{3.32}$$

Apparently all generators of level $a > N/2$ are null.

3.6 Truncation of a Nambu-Poisson algebra

While Nambu-Poisson algebras are always Lie 3-algebras of infinite dimensions, it is sometimes possible to truncate the Nambu-Poisson algebra to a finite dimensional Lie 3-algebra. We have seen such an example in section 3.4. In fact, the same can be done for all linear Nambu-Poisson algebras. Starting with a linear Nambu-Poisson algebra, one can impose a truncation over monomials of the coordinates of order larger than N. The reason why this is a consistent truncation for the Nambu bracket is essentially the same as the arguments in section 3.4.

3.7 Level extension of a 3-algebra

In the above we have seen that the notion of an additive level can be introduced to extend a given 3-algebra to a larger algebra. More precisely, given a 3-algebra

$$[T_i, T_j, T_k] = f_{ijk}{}^l T_l, \tag{3.33}$$

with an invariant metric h_{ij}, we can define a new 3-algebra for generators $T_i^{(a)}$ ($a = N_1, \cdots, N_2$ with $N_1 \geq 0$)

$$[T_i^{(a)}, T_j^{(b)}, T_k^{(c)}] = f_{ijk}{}^l T_l^{(a+b+c)}. \tag{3.34}$$

When $N_1 = 0$ the original 3-algebra is embedded at level 0.

A nontrivial choice of the metric is

$$\langle T_i^{(a)}, T_j^{(b)} \rangle = f(a+b) h_{ij}, \tag{3.35}$$

for an arbitrary function $f(a)$ such that

$$f(a) = 0 \quad \text{for} \quad a > N_1 + N_2. \tag{3.36}$$

To check that this is invariant, we note that

$$\langle [T_i^{(a)}, T_j^{(b)}, T_k^{(c)}], T_l^{(d)} \rangle + \langle T_k^{(c)}, [T_i^{(a)}, T_j^{(b)}, T_l^{(d)}] \rangle = (f_{ijk}{}^m h_{ml} + f_{ijl}{}^m h_{mk}) f(a+b+c+d) = 0, \tag{3.37}$$

whenever there is no truncation in both terms. When there is a truncation, we either have $a+b+c > N_2$ or $a+b+d > N_2$. This implies that $a+b+c+d > N_1 + N_2$, and the equality above still holds because $f(a+b+c+d) = 0$.

This is not the most general solution for the invariant metric. While generators $T_i^{(a)}$ at level $a < 3N_1$ can never appear on the right hand side of a Nambu bracket, it is impossible to write down any constraint for the metric components $\langle T_i^{(a)}, T_j^{(b)} \rangle$ with $a, b < 3N_1$. Those components are thus arbitrary.

3.8 A conjecture

The reason why examples of 3-algebra are so rare can be intuitively understood by noting the resemblance between the fundamental identity and the Plücker relation when a positive-definite metric is assumed. In the appendix we give a more detailed analysis of the fundamental identity with an effort to make its connection to the Plücker relation more manifest. We hope this will help us understand the fundamental identity better in the future.

In [21] it was conjectured that an n-algebra is always a direct product of n-algebras of dimension n and $(n+1)$ and some trivial algebras. This conjecture is ruled out by some of the examples listed above. On the other hand, except $\mathcal{A}_4$ and the trivial algebra (and their direct products), none of the examples we have so far has a metric which is positive definite. All of them have generators of zero-norm. Hence we conjecture that *all finite dimensional 3-algebras with positive-definite metrics are direct products of $\mathcal{A}_4$ with trivial algebras*. In other words, *except direct products of $\mathcal{A}_4$ with trivial algebras, all finite dimensional 3-algebras have generators of zero-norm.*

A weaker form of the conjecture has already been studied in [22]. There it was shown that nontrivial finite-dimensional generalization of $\mathcal{A}_4$, which is associated to the Lie algebra $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$, to other semi-simple Lie algebras is essentially impossible.

For an algebra with a positive-definite metric, we can always choose a new basis of generators such that the metric is the identity matrix δ_{ab}. It follows from the invariance of the metric

$$\langle [T_a, T_b, T_c], T_d \rangle + \langle T_c, [T_a, T_b, T_d] \rangle = 0 \tag{3.38}$$

that

$$f_{abcd} = -f_{abdc} \qquad (f_{abcd} \equiv f_{abc}{}^e h_{ed}). \tag{3.39}$$

Since the structure constants are by definition skew-symmetric with respect to the first 3 indices, in this case the 3-algebra structure constants are totally-antisymmetrized.

Assuming that the structure constants are totally-antisymmetrized, we checked using computers that all 3-algebras with no more than 8 generators are either trivial or are a direct product of the 4-generator algebra $\mathcal{A}_4$ with a trivial algebra.

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The almost unavoidable appearance of the zero-norm (or null) generators is very interesting from the viewpoint of physical applications. For a dynamical variable X living in the space of a 3-algebra with generators $\{T_A\}$,

$$X = X_A T_A, \tag{3.40}$$

its canonical kinetic term

$$\langle \partial_\mu X, \partial_\mu X \rangle \tag{3.41}$$

there is no quadratic term for X_A if T_A is a null generator. Hence the degrees of freedom associated with the zero-norm generators are not dynamical. They can be integrated out and their equations of motion are constraints. Therefore, each zero-norm generator corresponds to a gauge symmetry. Similarly, a negative norm generator corresponds to a ghost.

Infinite dimensional algebras with positive definite metrics are easy to construct. As we mentioned in section 2.2, for any Nambu-Poisson structure on the algebra $C(\mathcal{M}_3)$ of functions on a 3-dimensional space $\mathcal{M}_3$, the Nambu-Poisson tensor field defines a volume form on $\mathcal{M}_3$, which can be used to define an integral and then a metric. Whenever the volume form is everywhere non-vanishing, this metric is positive definite.

4. Representations of Nambu bracket by cubic matrix

4.1 Motivation

We would like to study representations of the Lie 3-algebra in this section. The first question is whether it is possible to represent the generators as matrices, which form an associative algebra. A natural definition of the quantum Nambu bracket is [1, 13]

$$[A, B, C] = ABC - ACB + BCA - BAC + CAB - CBA \tag{4.1}$$

for an associative algebra with elements A, B, C. For the algebra $\mathcal{A}_4$, there are representations of arbitrary dimension $N \geq 2$ [8] based on the $N \times N$ irreducible representation of $su(2)$. Let J^i $(i = 1, 2, 3)$ be the $N = 2j+1$ dimensional irreducible representation of $su(2)$, then

$$R(T^i) = \frac{1}{(j(j+1))^{1/4}} J^i, \qquad R(T^4) = (j(j+1))^{1/4} I, \tag{4.2}$$

where $i = 1, 2, 3$ and I is the unit matrix, is a representation of $\mathcal{A}_4$.

A problem with this representation is that the eigenvalues of $R(T^4)$ are fully degenerate. Interpreting $R(T^i)$ as some sort of quantum coordinates of $\mathbb{R}^4$, the geometric picture of this algebra is a fuzzy 2-sphere embedded in $\mathbb{R}^4$, with its 4-th coordinate fixed by

$$x^4 = (j(j+1))^{1/4}. \tag{4.3}$$

On the other hand, in the physical applications we have in mind, one would like to interpret $\mathcal{A}_4$ as a fuzzy 3-sphere.

Formally, $\mathcal{A}_4$ is a generalization of $su(2)$. While the adjoint representation of $su(2)$ is

$$(J_i)_{jk} = \epsilon_{ijk}, \tag{4.4}$$

one is tempted to conjecture that for $\mathcal{A}_4$ we have a representation of the form

$$R(T^i)_{jkl} \sim \epsilon_{ijkl}. \tag{4.5}$$

This is not exactly correct but we do have a representation of a similar form, which will be given below in (4.14). The point here is that although our lives would be much easier if we could just use matrices to represent Lie 3-algebras, but for the example of $\mathcal{A}_4$, it seems more appropriate to use objects with 3 indices.

There is also some physical motivation suggesting the use of cubic matrices. A long-standing puzzle about the low energy theory of coincident M5-branes is the following. In analogy with the case of D-branes, we imagine that cylindrical open membranes stretched between 2 M5-branes account for the low energy fields on M5-branes, and thus the low energy effective theory of N M5-branes is expected to be a non-Abelian gauge theory with N^2 degrees of freedom. On the other hand, anomaly and entropy computations suggest that the M5-brane world-volume theory has N^3 degrees of freedom [25]. Recently, arguments were presented based on considerations of membrane scattering amplitudes in the large C limit, suggesting that the dominating configuration of membranes connecting M5-branes is not a cylindrical M2-brane stretched between 2 M5-branes, but rather a triangular M2-brane stretched among 3 M5-branes [23]. The low energy fields on M5-branes should hence appear as objects with 3 indices. As a supporting evidence, BPS configurations of membranes stretched among 3 M5-branes were found in [26]. Therefore it is natural to introduce cubic matrices $X^i_{\alpha\beta\gamma}$, $i = 1, 2, 3, 4$ and $\alpha, \beta, \gamma = 1, \ldots, N$, to represent the spatial coordinates of open membranes with boundaries divided into 3 sections belonging to 3 M5-branes $(\alpha\beta\gamma)$.

4.2 Realization by cubic matrices

Cubic matrices were introduced in [7, 8]. A cubic matrix is an object with 3 cyclic indices

$$A_{ijk} = A_{jki} = A_{kij}. \tag{4.6}$$

A triplet product of cubic matrices is defined as

$$(A, B, C)_{ijk} = \sum_l A_{lij} B_{lki} C_{ljk}. \tag{4.7}$$

While Einstein's summation convention sums over indices repeated twice, we will only sum over indices repeated thrice.[7] The Hermitian conjugation is defined by

$$A^\dagger_{ijk} = A^*_{kji}, \tag{4.8}$$

[7]Because of this property, this triplet product is not invariant under the rotation (or the unitary transformation) of the indices. It motivates us to introduce a generalized product in section 4.5.

and the inner product of two cubic matrices by

$$\langle A|B\rangle \equiv \sum_{ijk} A^*_{ijk} B_{ijk}. \tag{4.9}$$

Note that we used slightly different notations for the inner product for cubic matrices $\langle\cdot|\cdot\rangle$ and the inner product for 3-algebra $\langle\cdot,\cdot\rangle$.

The cubic matrix algebra has some interesting properties. For example, it can be used to give a formulation of the generalized uncertainty relation for 3 observables [8]. The algebra of cubic matrix also naturally arises when we consider the scattering of open membranes in a large C field background [23].

The Nambu bracket is defined for cubic matrices as

$$[A,B,C] = (A,B,C) + (B,C,A) + (C,A,B) - (C,B,A) - (B,A,C) - (A,C,B). \tag{4.10}$$

4.3 Representations for $\mathcal{A}_4$

The algebra $\mathcal{A}_4$ (2.12) has been studied in the context of cubic matrices as the "generalized spin algebra" [8].

A $4\times 4\times 4$ representation of the algebra (2.12) is

$$R(T^i)^{jkl} = \begin{cases} e^{i\Omega^i_{jkl}} & \text{for} \quad i \neq j \neq k \neq l; \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

Ω^i_{jkl} is anti-symmetric $\Omega^i_{jkl} = -\Omega^i_{kjl}$, and cyclic $\Omega^i_{jkl} = \Omega^i_{klj}$. They satisfy

$$\Omega^i_{jkl} - \Omega^j_{kli} + \Omega^k_{lij} - \Omega^l_{ijk} = \frac{\pi}{2}\epsilon_{ijkl}. \tag{4.12}$$

The sign of each term corresponds to the orientation of a face of a tetrahedron. One way to assign values to Ω's is

$$\Omega^i_{jkl} = \frac{\pi}{8}\epsilon_{ijkl}. \tag{4.13}$$

In this case (4.11) can be expressed as

$$R(T^i)^{jkl} = |\epsilon_{ijkl}| e^{i\epsilon_{ijkl}\pi/8}. \tag{4.14}$$

Obviously $R(T_i)$'s are all Hermitian.

This representation R has

$$\sum_{klm} R(T^k)_{lmi} R(T^k)_{lmj} = 3!\delta_{ij}, \tag{4.15}$$

which can be viewed as the analogue of the condition

$$\sum_{i=1}^{4} X_i^2 = r^2 \tag{4.16}$$

that defines a 3-sphere of radius r in $\mathbb{R}^4$. Therefore it is natural to associate $\mathcal{A}_4$ to the notion of a fuzzy 3-sphere. Note that this algebra is different from the definition of fuzzy 3-sphere in [24].

Representations of arbitrary dimension $N > 4$ can be found in [8].

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4.4 Construction of higher representations

Here we would like to discuss a question about cubic matrix representations for a generic Lie 3-algebras, that is, how to construct new representations from given representations. Like the representation by matrices, it is possible to construct higher dimensional representations by the direct sum and the direct product for the representation by cubic matrices.

Suppose $R_i(T^a)$ $(i = 1, 2)$ is an N_i dimensional cubic matrix which satisfies a given 3-algebra (not necessarily $\mathcal{A}_4$). There are several systematic ways to construct new cubic matrix representations of the same 3-algebras from R_i:

1. Direct sum representation $R_1 \oplus R_2$ $(N_1 + N_2$ dim):

$$(R_1 \oplus R_2(T^a))_{ijk} = \begin{cases} R_1(T^a)_{ijk} & \text{if } i,j,k \in \{1,\cdots,N_1\}\,, \\ R_2(T^a)_{i-N_1,j-N_1,k-N_1} & \text{if } i,j,k \in \{N_1+1,\cdots,N_1+N_2\}\,, \\ 0 & \text{otherwise.} \end{cases} \quad (4.17)$$

2. Direct product representations $R_1 \otimes R_2$ which has dimension $N_1 N_2$:

$$(R_1 \otimes R_2)_{IJK} = (R_1(T^a))_{ijk}\delta_{i'j'k'} \pm \delta_{ijk}(R_2(T^a))_{i'j'k'}, \qquad \delta_{ijk} := \delta_{ij}\delta_{ik}\,. \quad (4.18)$$

   Here I, J, K is the combination of two indices such as $I = (i, i')$, $J = (j, j')$, $K = (k, k')$. i, j, k are in $1, \cdots, N_1$ and i', j', k' are in $1, \cdots, N_2$. We can take both sign in the second term since $-R_2(T^a)$ is also the representation of the 3-algebra.

3. Tensor product $R(T^a) \otimes \mathcal{Z}$ with constant cubic matrix $\mathcal{Z}$ which satisfies

$$(\mathcal{Z}, \mathcal{Z}, \mathcal{Z}) = \mathcal{Z}. \quad (4.19)$$

   If the size of $\mathcal{Z}$ is $n \times n \times n$, the dimension of the representation is nN. There are many choices of $\mathcal{Z}$. Somewhat systematic construction of $\mathcal{Z}$ is given later.

   By taking the direct product of the fundamental representation of $\mathcal{A}_4$, one can obtain 4^n dimensional representations systematically.

   In the representation theory of matrices, one may use the unitary transformation by which the representation matrix becomes block diagonal form. This notion, however, does not have straightforward generalization to the cubic matrices.

Construction of cubic projector $\mathcal{Z}$. Straightforward solutions of (4.19) are the diagonal cubic matrices,

$$Z_{ijk} = z_i \delta_{ijk}, \qquad z_i = \pm 1, 0\,. \quad (4.20)$$

For less trivial solutions, we observe that eq. (4.19) resembles the projector equation. It motivates us seek solutions of the form,

$$\mathcal{Z}_{ijk} = v_i v_j v_k \quad (4.21)$$

where v_i is an vector in n dim space.

By requiring eq. (4.19), we obtain,

$$(\mathcal{Z}, \mathcal{Z}, \mathcal{Z})_{ijk} = \left(\sum_l v_l^3\right) v_i^2 v_j^2 v_k^2 \,. \tag{4.22}$$

So if

$$v_i^2 = \left(\sum_l v_l^3\right)^{-1/3} v_i, \tag{4.23}$$

(4.21) gives a solution to (4.19). The general solution to this is

$$v_i = c\epsilon_j, \qquad \epsilon_j = \pm 1, 0, \tag{4.24}$$

$$c = \left(\sum_i \epsilon_i\right)^{-1/6} . \tag{4.25}$$

This construction can be generalized by using $r(< n)$ vectors $v_i^{(\alpha)}$ $(\alpha = 1, \cdots, r)$, where each $v^{(\alpha)}$ takes the form (4.24) and the cubic orthogonality relation,

$$\sum_i v_i^{(\alpha)} v_i^{(\beta)} v_i^{(\gamma)} \propto \delta_{\alpha\beta\gamma}. \tag{4.26}$$

Then,

$$\mathcal{Z}_{ijk} = \sum_{\alpha=1}^{r} \mathcal{Z}_{ijk}^{(\alpha)},$$

$$\mathcal{Z}_{ijk}^{(\alpha)} = v_i^{(\alpha)} v_j^{(\alpha)} v_k^{(\alpha)} \tag{4.27}$$

$$(\mathcal{Z}^{(\alpha)}, \mathcal{Z}^{(\beta)}, \mathcal{Z}^{(\gamma)}) = \begin{cases} \mathcal{Z}^{(\alpha)} & \text{if } \alpha = \beta = \gamma \\ 0 & \text{otherwize} \end{cases} \tag{4.28}$$

satisfies (4.19). One might refer to such $\mathcal{Z}$ as rank r cubic projector.

We note that this construction does not give all the cubic projectors. Even for the $2 \times 2 \times 2$ case, a direct algebraic computation by computer shows that there are extra solutions which do not take this form

4.5 Comments on irreducibility

As mentioned earlier, the non-invariance of the triplet product (4.7) under the rotation of the indices forces us to introduce a generalization of the product by using a symmetric cubic matrix $\mathcal{K}$, $(\mathcal{K}_{i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)}} = \mathcal{K}_{i_1 i_2 i_3})$,

$$(A, B, C)_{ijk} = \sum_{n,m,l,i',i'',j',j'',k',k''} \mathcal{K}_{nml} A_{ni'j''} B_{mk'i''} C_{lj'k''} \mathcal{K}_{ii'i''} \mathcal{K}_{jj'j''} \mathcal{K}_{kk'k''} \tag{4.29}$$

where the indices i, j, k, n run from 1 to N. Usually we take $\mathcal{K}_{ijk} = \delta_{ijk}$. We note that there is no orthogonal transformation which keeps δ_{ijk} invariant. In the general form above, the summations are taken only for doubly repeated indices, so the notion of the orthogonal transformation remains the same.

Suppose we consider a triplet product algebra such as $[J^a, J^b, J^c] = i\epsilon_{a,b,c,d}J^d$, ($J^a := R(T^a)$) and try to find "irreducible decomposition". We introduce the orthogonal projectors p_{ij} and q_{ij} which satisfy

$$p^2 = p, \quad q^2 = q, \quad p^t = p, \quad q^t = q, \quad pq = 0, \quad p + q = 1\,. \tag{4.30}$$

We note that such projector may be written as,

$$p = g \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} g^t\,, \quad q = g \begin{pmatrix} 0 & 0 \\ 0 & I_{N-d} \end{pmatrix} g^t\,, \quad g \in O(N, \mathbf{R}) \tag{4.31}$$

One may define the algebra be reducible if there exists a pair p, q as above and they satisfy

$$\sum_{ij} (J^a)_{ijk} p_{ii'} q_{jj'} = \sum_{jk} (J^a)_{ijk} p_{jj'} q_{kk'} = \sum_{ij} (J^a)_{ijk} p_{kk'} q_{ii'} = 0\,, \tag{4.32}$$

$$\sum_{ij} (\mathcal{K})_{ijk} p_{ii'} q_{jj'} = \sum_{jk} (\mathcal{K})_{ijk} p_{jj'} q_{kk'} = \sum_{ki} (\mathcal{K})_{ijk} p_{kk'} q_{ii'} = 0\,. \tag{4.33}$$

If these identities are satisfied, we have a d dimensional representation by redefining the generators and the cubic product at the same time as

$$J^a \to (\tilde{J}^a)_{ijk} = \sum_{i'j'k'} (J^a)_{i'ij'k'} p_{i'i} p_{j'j} p_{k'k}, \tag{4.34}$$

$$\mathcal{K} \to (\tilde{\mathcal{K}})_{ijk} = \sum_{i'j'k'} (\mathcal{K})_{i'ij'k'} p_{i'i} p_{j'j} p_{k'k}. \tag{4.35}$$

An example of reducible representation For a given representation J^a, the representation $\mathcal{J}^a = J^a \otimes \mathcal{Z}$, where $\mathcal{Z}_{\bar{\imath}\bar{\jmath}\bar{k}}$ is written as (4.21), gives an example of the reducible representation. The projectors are,

$$p_{IJ} = \delta_{ij} \frac{v_{\bar{\imath}} v_{\bar{\jmath}}}{\sqrt{|v|^2}}\,, \quad q_{IJ} = \delta_{ij} \left(1 - \frac{v_{\bar{\imath}} v_{\bar{\jmath}}}{\sqrt{|v|^2}} \right)\,. \tag{4.36}$$

In this sense, the tensor product with the cubic projector gives a good example of the reducible representation in our sense. We note, however, that the cubic matrices $\mathcal{K}$ which defines the cubic product is not given by the original definition δ_{ijk} because of eq. (4.35).

Failed example: (anti-)symmetrization In case of the Lie algebra, the tensor product of two fundamental representations are reducible. Reduction to the irreducible representation can be obtained by using (anti-)symmetrization of indices. In the following, We will argue that this will not be so simple for the cubic case.

We consider a direct product representation of two fundamental representations,

$$J^a_{IJK} = J^a_{ijk} \delta_{\bar{\imath}\bar{\jmath}\bar{k}} + J^a_{\bar{\imath}\bar{\jmath}\bar{k}} \delta_{ijk} \tag{4.37}$$

and $\mathcal{K}_{IJK} = \delta_{ijk}\delta_{\bar{\imath}\bar{\jmath}\bar{k}}$. Here we use the multi-indices I, J, K to represent $i, \bar{\imath}$ and so on.

We define the projections to the symmetric and anti-symmetric part as

$$p_{IJ} = \frac{1}{2}\left(\delta_{ij}\delta_{\bar{\imath}\bar{\jmath}} + \delta_{i\bar{\jmath}}\delta_{\bar{\imath}j}\right), \quad q_{IJ} = \frac{1}{2}\left(\delta_{ij}\delta_{\bar{\imath}\bar{\jmath}} - \delta_{i\bar{\jmath}}\delta_{\bar{\imath}j}\right). \tag{4.38}$$

It is easy to see that p, q satisfy the constraint (4.30). On the other hand, conditions (4.32)–(4.33) become

$$\begin{aligned}\sum_{IJ} J^a_{IJK} p_{IL} q_{JM} &= \frac{1}{4}\left(J^a_{lmk}\delta_{\bar{l}\bar{m}\bar{k}} - J^a_{l\bar{m}k}\delta_{\bar{l}m\bar{k}} + J^a_{\bar{l}mk}\delta_{l\bar{m}\bar{k}} - J^a_{\bar{l}\bar{m}k}\delta_{lm\bar{k}}\right.\\ &\qquad \left. +\delta_{lmk}J^a_{\bar{l}\bar{m}\bar{k}} - \delta_{l\bar{m}k}J^a_{\bar{l}m\bar{k}} + \delta_{\bar{l}mk}J^a_{l\bar{m}\bar{k}} - \delta_{\bar{l}\bar{m}k}J^a_{lm\bar{k}}\right) \qquad (4.39)\\ \sum_{IJ} \delta_{IJK} p_{IL} q_{JM} &= \frac{1}{2}\left(\delta_{lmk}\delta_{\bar{l}\bar{m}\bar{k}} - \delta_{l\bar{m}k}\delta_{\bar{l}m\bar{k}} + \delta_{\bar{l}mk}\delta_{l\bar{m}\bar{k}} - \delta_{\bar{l}\bar{m}k}\delta_{lm\bar{k}}\right) \qquad (4.40)\end{aligned}$$

They do not vanish. It implies that the (anti-)symmetrization which works in the construction of the representation of Lie algebra does not work for cubic matrices.

5. Application to multiple M2-branes

5.1 Basu-Harvey equation

Generalizing Nahm's equation, which was used to describe the analogous configuration of D1-branes ending on D3-branes, the Basu-Harvey equation was proposed [27] to describe multiple M2-branes ending on an M5-brane

$$\frac{dX^i}{ds} + i\frac{K}{3!}\epsilon^{ijkl}[X^j, X^k, X^l] = 0, \tag{5.1}$$

where $X^i(s)$'s represent spatial fluctuations of the M2-branes, and s is a worldvolume coordinate. This equation admits a funnel solution:

$$X^i(s) = f(s)R(T^i), \tag{5.2}$$

$$f(s) = \frac{1}{\sqrt{2Ks}}, \tag{5.3}$$

where T^i satisfies the SO(4)-invariant algebra $\mathcal{A}_4$

$$[T^i, T^j, T^k] = i\epsilon^{ijkl}T^l, \qquad (i, j, k, l = 1, 2, 3, 4,) \tag{5.4}$$

and $R(T^i)$ is any representation of this algebra.

As we will see below, the Basu-Harvey equation can be interpreted as a BPS condition for the multiple M2-brane action of Bagger and Lambert [5], although it was first proposed without an underlying Lagrangian. On the other hand, this particular solution happens to define a Lie 3-algebra structure. It is possible to proceed for our present purpose without assuming a particular M2-brane action.

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In order to give a proper geometrical interpretation to this solution, we also need to assume that the algebra (5.4) of T^i describes a *fuzzy three-sphere* with radius r given by

$$r^2 \equiv \sum_i (X^i)^2 \propto f^2(s) \propto \frac{1}{Ks}. \tag{5.5}$$

Hence

$$r^2 = \frac{\alpha}{Ks} \tag{5.6}$$

for some constant α. The T^i's then represent the Cartesian coordinates of the fuzzy 3-sphere. Furthremore, infinitesimal SO(4) rotations are generated by

$$\delta T^k = \Lambda_{ij}[T^i, T^j, T^k], \tag{5.7}$$

and the invariant metric is

$$\langle T^i, T^j \rangle = \delta^{ij}. \tag{5.8}$$

The energy proposed in [27] is

$$\begin{aligned} E = T_2 N \int d^2\sigma \Bigg[a^2 \left| \frac{dX^i}{ds} - i\frac{K}{3!}\epsilon^{ijkl}[X^j, X^k, X^l] \right|^2 \\ + \left(1 + i\frac{C}{3!}\epsilon^{ijkl} \left\langle \frac{dX^i}{ds} \middle| [X^j, X^k, X^l] \right\rangle \right)^2 \Bigg]^{1/2}, \end{aligned} \tag{5.9}$$

where $|A|^2 \equiv \langle A|A\rangle$. We will specify the two constant parameters a and C below.

For $X^i = 0$ (or more generally when $\frac{dX^i}{ds} = 0 = [X^j, X^k, X^l]$), the energy is that of N D2-branes at rest: $E = T_2 N$ times the M2-brane volume. The form of the energy E is such that the Basu-Harvey equation (5.1) is a BPS condition. One should choose a as

$$a^2 = \frac{C}{K} \tag{5.10}$$

so that the cross-term proportional to $\langle \frac{dX^i}{ds} | [X^j, X^k, X^l]\rangle$ cancels in (5.9), otherwise the theory is not covariant.

For the funnel solution (5.2) and (5.3), the energy is

$$E = T_2 N \int d^2\sigma \left| 1 + \frac{C}{K} \left\langle \frac{dX^i}{ds} \middle| \frac{dX^i}{ds} \right\rangle \right| = T_2 N L \int ds + T_2 N L \int ds \frac{C}{8K^2 s^3} \langle R(G^i)|R(G^i)\rangle. \tag{5.11}$$

According to (5.6),

$$\int_0^\infty \frac{ds}{s^3} = \frac{2K^2}{\alpha^2} \int_0^\infty dr\, r^3, \tag{5.12}$$

and thus

$$E/L = T_2 N \int ds + \beta \int dr\, r^3, \tag{5.13}$$

where

$$\beta = 2T_2 N \frac{\langle R(G^i)|R(G^i)\rangle}{8\alpha^2} C. \tag{5.14}$$

We should choose C such that

$$\beta = 2\pi^2 T_5, \tag{5.15}$$

where T_5 is the M5-brane tension.

The derivation above goes through without the need of a representation for the bracket in (5.1). While the constant C can be tuned to give the correct answer, the needed r^3 dependence of the 2nd term in E is also guaranteed by the relation (5.6)

$$r^2 \propto \frac{1}{s}, \tag{5.16}$$

which is a direct result of the fact that the two terms in the Basu-Harvey equation differ in the order of X by 2.

After choosing C properly to get the correct expression of energy for the M2-M5 system, K is still a free parameter. But we can always scale X so that $K = 1$.

In the original work of Basu and Harvey [27], they considered the fuzzy 3-sphere defined in [24]. What we have shown above is that actually the success of Basu-Harvey equation does not rely on a particular choice of how the fuzzy 3-sphere algebra (5.4) is realized. All we need are the general properties of the Lie 3-algebra.

5.2 Multiple M2-brane action

Bagger and Lambert [3–5] proposed a supersymmetric Lagrangian for M2-branes for a given 3-algebra as

$$\mathcal{L} = -\frac{1}{2}\langle D^\mu X^I, D_\mu X^I\rangle + \frac{i}{2}\langle\bar\Psi, \Gamma^\mu D_\mu \Psi\rangle + \frac{i}{4}\langle\bar\Psi, \Gamma_{IJ}[X^I, X^J, \Psi]\rangle - V(X) + \mathcal{L}_{CS}, \tag{5.17}$$

where D_μ is the covariant derivative, $V(X)$ is the potential term defined by

$$V(X) = \frac{1}{12}\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\rangle, \tag{5.18}$$

and the Chern-Simons action for the gauge potential is

$$\mathcal{L}_{CS} = \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}\right). \tag{5.19}$$

The SUSY transformation is defined by

$$\delta X^I_a = i\bar\epsilon\Gamma^I\Psi_a, \tag{5.20}$$

$$\delta\Psi_a = D_\mu X^I_a \Gamma^\mu\Gamma^I\epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\epsilon, \tag{5.21}$$

$$\delta\tilde{A}_\mu{}^b{}_a = i\bar\epsilon\Gamma_\mu\Gamma_I X^I_c \Psi_d f^{cdb}{}_a. \tag{5.22}$$

While the fundamental identity is needed for the gauge symmetry of the multiple M2-brane theory, the invariant metric is also necessary to write down the gauge-invariant Lagrangian.

For the background with $\Psi = \tilde{A} = 0$, a BPS condition should guarantee that

$$\left(\partial_\mu X^I \Gamma^\mu \Gamma^I - \frac{1}{6}[X^I, X^J, X^K]\Gamma^{IJK}\right)\epsilon = 0 \tag{5.23}$$

for some constant spinor ϵ. Assuming that $\partial_t = \partial_\sigma = 0$, for the constant spinor satisfying

$$\left(1 + \frac{i}{K}\Gamma^s\Gamma^{1234}\right)\epsilon = 0, \tag{5.24}$$

the BPS condition is guaranteed if

$$\frac{dX^i}{ds} + i\frac{K}{3!}\epsilon^{ijkl}[X^j, X^k, X^l] = 0, \tag{5.25}$$

where the superscript s on Γ^s denotes the direction in which X^s is identified with the M2-brane worldvolume coordinate s, and $\Gamma^{1234} \equiv \Gamma^1\Gamma^2\Gamma^3\Gamma^4$, and we also assumed that $X^I = 0$ except for $I = 1, 2, 3, 4$. We see that the Basu-Harvey equation is indeed a BPS condition for this theory if $K = \pm 1$ (this can always be achieved by scaling X).

For a solution of the Basu-Harvey equation, the Hamiltonian density of the Bagger-Lambert model is simply

$$\mathcal{H} = \langle \partial_s X^I, \partial_s X^I \rangle. \tag{5.26}$$

This coincides with the Hamiltonian proposed in [27] up to a constant shift and overall factor.

Although the the connection between the Basu-Harvey equation and the Bagger-Lambert model begins to be clarified we have an impression that there still remain some mysteries which should be clarified in the future. Incidentally, apart from the Basu-Harvey equation, the study of Bagger-Lambert model with boundaries [28] is another approach to M5-branes from the M2-brane viewpoint.

6. Comments

6.1 Lie 3-Algebra

In this paper we discussed quite a few new examples of Lie 3-algebra of finite dimensions. Yet we still have the basic problem of lacking any mathematical structure analogous to the matrix algebra, which guarantees that the commutator defines a Lie algebra. The fundamental identity appears to be much more restrictive than the Jacobi identity, and we do not know much about how to solve it.

The truncation of a Nambu-Poisson bracket (sections 3.4, 3.6) can be used to construct a finite dimensional Lie 3-algebra. While the naive truncation works well, it will be desirable to find a deformed truncation such that the final 3-algebra possesses better properties. A possible motivation is to avoid negative norm generators in the algebra. Another example is that, for the truncated Nambu bracket on S^3, the radius constraint $x_i^2 = r^2$ can not be imposed until computing the metric. Although the linear dependence among functions will be fixed by the metric, and thus this will only result in some redundancy of the generators, similar to what happens when we use an over-complete basis of functions on a manifold, it would be better if this 3-algebra can be deformed such that the constraint can be imposed directly on the generators.

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One can apply the general procedures of section 3.7 to a given 3-algebra for an arbitrary number of times to obtain more and more new examples of Lie 3-algebras. Yet it remains to be seen how nontrivial these examples will be.

For physical applications to multiple M2-branes, since we want the M2-branes turn into D2-branes upon compactifying a spatial direction, we hope to associate the $su(N)$ Lie algebra with a Lie 3-algebra for each N. So far we only know that $\mathcal{A}_2$ is associated with $su(2)$ [29]. In section 3.3, we present a 3-algebra based on an arbitrary Lie algebra. However its metric is almost trivial. It is most desirable to find Lie 3-algebras associated to all $su(N)$'s.

6.2 Cubic matrices

There are a few issues regarding cubic matrices which should be studied further in the future.

First, in the construction of higher representations, we introduced the direct product. In case of Lie algebra, such a procedure produces *reducible* representations and we have to decompose them to extract the irreducible representations. In order to do similar reduction, we need to define the corresponding notions of the direct sum representations and the unitary equivalence between representations, i.e., representations R and R' are equivalent if there exists a unitary matrix U such that $R'(T) = UR(T)U^\dagger$. For cubic-matrix representations, it is trivial to see that the direct sum gives a new representation. On the other hand, in order to define the unitary equivalence, it is natural to use the Nambu bracket $\delta R = [R, K_1, K_2]$, for some K_1 and K_2, and we need to impose the fundamental identity in order to preserve the algebraic structure. However, the fundamental identity is not satisfied for generic elements of the cubic matrices. The subset of cubic matrices which is known to satisfy the fundamental identity is the set of objects called "normal matrices" [7]. They are, however, an analogue of diagonal matrices and give rise to a trivial change of the representation.

Second, in this paper, we introduce only the triplet multiplication (4.7). By composing it, we can generate functions of odd power. This is not sufficient to produce all functions on a fuzzy space to guarantee a proper classical limit. In order to generate a generic function, we would need other type of products. As we commented in our previous paper [23], for such a direction, it will be necessary to introduce objects with more indices $\Psi_{i_1\cdots i_n}$. How to construct a series of the products consistently remains a big challenge.

6.3 Multiple M2-branes

Recently there is a very interesting paper [29] which proposed a novel Higgs mechanism for the Bagger-Lambert model [3] so that the multiple M2-brane action reduces to the D2-brane effective action upon compactification of a spatial coordinate. Later it was realized that [30 − 32] the moduli space for the model with the $\mathcal{A}_4$ algebra does not match with the moduli space for 2 M2-branes in flat space, but rather it matches with the moduli space for an orbifold. While this is a success of the Bagger-Lambert model, it is now even more urgent to consider more examples of 3-algebras for the Bagger-Lambert model to go beyond a single special case. It will be very interesting to see whether some of the examples provided in this work will correspond to a certain physical background for M2-branes in M theory. It will also be very intriguing to find out the physical interpretation of the ubiquitous zero-norm generators. In some of the examples there are also negative norm generators, which can potentially result in ghosts in the model. Perhaps those algebras with negative norm generators should be dismissed in certain applications, just like we usually avoid non-compact Lie groups in certain physical problems. It will be interesting to see whether there are other physical applications of the Lie 3-algebra besides M2-branes physics.

Note added. After we submitted this paper to arXiv, we are informed that the relation between the fundamental identity and the Plücker relation was studied in [33] where a systematic study fundamental identity in $\mathcal{D} = 5, 6, 7, 8$ was also carried out.

We note that there have been substantial developments [34, 35] on the conjecture in section 3.8 after this paper appeared on the arXiv.

Acknowledgments

We appreciate partial financial support from Japan-Taiwan Joint Research Program provided by Interchange Association (Japan) by which this collaboration is made possible.

The authors thank Hidetoshi Awata, David Berman, Kazuyuki Furuuchi, Takeo Inami, Hsien-chung Kao, Xue-Yen Lin, Darren Sheng-Yu Shih, and Wen-Yu Wen for helpful discussions. P.-M. H. is supported in part by the National Science Council, and the National Center for Theoretical Sciences, Taiwan, R.O.C. Y. M. is partially supported by Grant-in-Aid (#16540232) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

A. Relation with Plücker relation

Here we show that there is a direct relation between the fundamental identity and Plücker relation which characterizes the locus of the Grassmannian manifold. This bilinear relation appeared in a variety of context in the physical literature, such as the exactly solvable system (KP hierarchy etc.), free fermions on Riemann surface, topological string, matrix model and so on.[8] Although this relation itself is not new in mathematical literature (see for example [15]), it might shed a new light in the study of the fundamental identity (2.4).

To see the relation, we rewrite the structure constant by the metric, by lowering the upper index by the metric, $f_{a_1,\cdots,a_{p+1}} = f_{a_1,\cdots,a_p}{}^b h_{ba_{p+1}}$, which gives the rank $p+1$ anti-symmetric tensor. It can be identified as the coefficients of the $p+1$ vector by writing them with the wedge product of the orthonormal basis of n dimensional vector space $\mathbf{e}_1, \cdots, \mathbf{e}_n$,

$$|f\rangle = \sum_{a_1,\cdots,a_{p+1}} f_{a_1\cdots a_{p+1}} \mathbf{e}_{a_1} \wedge \cdots \wedge \mathbf{e}_{a_{p+1}} . \tag{A.1}$$

[8] Many of the works have the origin in the identification of Hirota's bilinear identity of the KP hierarchy with the Plücker relation. One of the original work is, E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in Proc. RIMS Symp. on Nonlinear Integrable Systems (Kyoto, 1981), eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).

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Plücker relation is a condition on the coefficient $f_{a_1 \cdots a_{p+1}}$ when the $(p+1)$ vector $|f\rangle$ is written in the form,

$$|f\rangle = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{p+1}\,, \qquad \mathbf{v}_a \in \mathbf{R}^n\,. \tag{A.2}$$

The requirement is given by a set of bilinear relations,

$$\sum_{k=1}^{p+2} (-1)^k f_{a_1,\cdots,a_p,b_k} f_{b_1,\cdots,b_{k-1},b_{k+1},\cdots,b_{p+2}} = 0 \tag{A.3}$$

where $(a_1,\cdots,a_p)$ and $(b_1,\cdots,b_{p+2})$ is the arbitrary number in $1,\cdots,n$. The fundamental identity is obtained from Plücker relation by putting $a_1 = b_1 = a$ and take the sum over a. Because of this procedure, the fundamental identity is a weaker condition than the Plücker relation.

In particular, when

$$f_{a_1,\cdots,a_{p+1}} = \begin{cases} \epsilon_{a_1,\cdots,a_{p+1}} & a_1,\cdots,a_{p+1} \in \{1,\cdots,p+1\} \\ 0 & \text{otherwise} \end{cases} \tag{A.4}$$

the $(p+1)$-vector becomes $|f\rangle = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p+1}$. Therefore, it satisfies the Plücker relation and the fundamental identity. We note that the direct sum of this p-algebra corresponds to the p vector of the form $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p+1} + \mathbf{e}_{p+2} \wedge \cdots \wedge \mathbf{e}_{2p+2} + \cdots$ which is definitely not of the form (A.2). In this sense, the fundamental identity allows a broader set of solutions than the Plücker relation.

In the application to the physics, it may be useful to rewrite these relations by free fermions. To define them, we consider space of p-vectors, $\mathcal{H}_p$, $(p = 0,1,\cdots,n)$ where base is spanned by exterior product of the basis, $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}$, $(i_1 < \cdots < i_p)$. On this p-vector space, we introduce "fermion" operators $\psi_i, \bar{\psi}_i$ $(i = 1,\cdots,n)$ as

$$\bar{\psi}_a(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}) = \mathbf{e}_a \wedge \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}\,, \tag{A.5}$$

$$\psi_a(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} \delta_{a i_k} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{k-1}} \wedge \mathbf{e}_{i_{k+1}} \cdots \wedge \mathbf{e}_{i_p}\,. \tag{A.6}$$

These operators satisfy standard anticommutation relations,

$$\begin{aligned} \{\psi_i, \bar{\psi}_j\} &= \delta_{ij}\,, \\ \{\psi_i, \psi_j\} &= \{\bar{\psi}_i, \bar{\psi}_j\} = 0\,. \end{aligned} \tag{A.7}$$

The Plücker relation and the fundamental identity is then written in terms of the fermions as,

$$\text{Plücker relation}: \quad \sum_{i=1}^{n} \psi_i |f\rangle \otimes \bar{\psi}_i |f\rangle = 0\,, \tag{A.8}$$

$$\text{Fundamental identity}: \quad \sum_{i,j=1}^{n} \psi_j \psi_i |f\rangle \otimes \psi_j \bar{\psi}_i |f\rangle = 0\,. \tag{A.9}$$

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